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Partitions of factorisations of parameter words

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Abstract

We study partitions of factorisations of parameter words. We characterise the discernables of these factorisations with respect to partitions under substitutions under ascending parameter words.

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1. Introduction

Let A be a finite set. Recall that a k-parameter word of length n over A is a word w of length n over the alphabet $A \cup \{\lambda_1, \ldots, \lambda_k\}$ where $\{\lambda_1, \ldots, \lambda_k\} \cap A = \emptyset$, such that each of the parameters λ_i occurs at least once in the word w. In the sequel, we assume the parameters λ_i to be linearly ordered by the indices i and that, for $1 \leqslant i < j \leqslant k$, the first occurrence of λ_i precedes the first occurrence of λ_j in w. The set of such words is denoted by $A\binom{n}{k}$. If $w \in A\binom{n}{k}$, we say that w is an *ascending* parameter word when, for $1 \leqslant i < j \leqslant k$, each occurrence of λ_i precedes each occurrence of λ_j in w. The set of ascending k-parameter words of length n over A is denoted by $A^{<}\binom{n}{k}$. If $w \in A\binom{m}{k}$ and $f \in A\binom{n}{m}$, we define $f \circ w \in A\binom{n}{k}$ to be the word obtained from f by replacing, for every $i \leqslant m$, each occurrence of λ_i in f by the ith letter in w. It is clear that $f \circ w$ will be ascending when f and w are.

We recall the following well-known theorem.

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Theorem A (Graham and Rothschild [2]). *For each k, m and r and each finite alphabet A, there exists a number N such that for every r-colouring χ of $A\binom{N}{k}$ there exists some $f \in A\binom{N}{m}$ such that $\chi(f \circ u_1) = \chi(f \circ u_2)$ for every u_1, $u_2 \in A\binom{m}{k}$.*

The proof in [2] as well as later proofs of this theorem, as in [3] for example, use a double induction. The upper bounds of N obtained from these proofs do not depend primitive recursively on $|A|$, k, m and r. We shall discuss a variant of Theorem A for which primitive recursive bounds can be found. In this variant, we shall study colourings χ, not only of $A\binom{N}{k}$ but of all possible algebraic factorisations of these words (see Theorem C below).

In order to formulate the results, we introduce some notation. If w is any word over $A \cup \{\lambda_1, \ldots, \lambda_k\}$ (not necessarily a k-parameter word), let w' be the subword (subsequence) of w which is obtained by omitting all the alphabet letters of w. Thus, for example, if $w = 1\lambda_1 1\lambda_1 1\lambda_1\lambda_2\lambda_1$ then $w' = \lambda_1^3\lambda_2\lambda_1$. We can write w' in the form $\prod_{j=1}^{t} \lambda_{i_j}^{a_j}$ where $a_j \geqslant 1$ for all j and $i_{j+1} \neq i_j$ for all $j < t$. Define the associated *pattern* $\pi(w)$ of w to be the word $\prod_{j=1}^{t} \lambda_{i_j}$. Note that $\pi(w)$ is itself a parameter word over $\{\lambda_1, \ldots, \lambda_k\}$ in case w is in $A\binom{m}{k}$. In the preceding example, $w' = \lambda_1^3\lambda_2\lambda_1$ so that $\pi(w) = \lambda_1\lambda_2\lambda_1$. Note that if $w \in A\binom{m}{k}$ and $f \in A^{<}\binom{N}{m}$ then $\pi(f \circ w) = \pi(w)$. It follows that the Graham–Rothschild theorem does not hold if f is required to be ascending. The most one could hope for is to find some ascending f such that $\chi(f \circ w)$ with $w \in A\binom{m}{k}$ depends on $\pi(w)$ only. The following theorem will follow from the main result.

Theorem B. *For each $m, k, r \in \mathbb{N}$ and for a finite alphabet A, there exists a natural number N that depends primitive recursively on m, k, r and $|A|$, such that for every r-colouring χ of $A\binom{N}{k}$ there exists some $f \in A^{<}\binom{N}{m}$ such that for $w_1, w_2 \in A\binom{m}{k}$ it is the case that $\chi(f \circ w_1) = \chi(f \circ w_2)$ whenever $\pi(w_1) = \pi(w_2)$.*

In order to formulate the main result, we define a *factorisation* of a parameter word $w \in A\binom{m}{k}$ as a sequence $\overline{w} = (w_1, \ldots, w_s)$ of subwords w_i of w such that $w = w_1 \ldots w_s$. By the *type* $\tau(\overline{w})$ of $\overline{w}$ we mean the sequence,

$$\tau(\overline{w}) = (\pi(w_1), (w_{21}, \pi(w_2)), \ldots, (w_{s1}, \pi(w_s))),$$

where w_{i1} is the first letter of the subword w_i, $i \geqslant 2$.

As in [1] we associate with any $f \in A^{<}\binom{N}{m}$ and factorisation $\overline{w} = (w_1, \ldots, w_s)$ of a word $w \in A\binom{m}{k}$, a factorisation $\overline{f} = (f_1, \ldots, f_s)$ of f where for $i \geqslant 2$ the first letter of f_i is $\lambda_{v(i)}$ with $v(i) = |w_1| + \cdots + |w_{i-1}| + 1$ and, moreover, the parameter $\lambda_{v(i)}$ occurs in no f_j for $j < i$. Then we set $f \star \overline{w} = (f_1 \circ w_1, \ldots, f_s \circ w_s)$. It follows that $\tau(f \star \overline{w}) = \tau(\overline{w})$. Indeed,

$$\begin{aligned}\tau(f \star \overline{w}) &= \tau(f_1 \circ w_1, \ldots, f_s \circ w_s)\\ &= (\pi(f_1 \circ w_1), ((f_2 \circ w_2)_1, \pi(f_2 \circ w_2)), \ldots, ((f_s \circ w_s)_1, \pi(f_s \circ w_s)))\end{aligned}$$

$$= (\pi(w_1), (w_{21}, \pi(w_2)), \ldots, (w_{s1}, \pi(w_s)))$$
$$= \tau(\overline{w}).$$

We can now formulate the main result.

Theorem C. *For $m, k, r \in \mathbb{N}$ and any finite alphabet A, there exists a natural number N that depends primitive recursively on m, k, r and $|A|$, such that for every r-colouring χ of the class of factorisations of words in $A\binom{N}{k}$ there exists some $f \in A^{<}\binom{N}{m}$ such that for all $u \in A\binom{m}{k}$ and all factorisations $\overline{u}$ of u, the colour $\chi(f \star \overline{u})$ depends on $\tau(\overline{u})$ only.*

It is clear that Theorem B is a special case of Theorem C. In the proof of Theorem C, we make use of Theorem B in [1] which is the version of Theorem C for ascending parameter words in $A^{<}\binom{N}{k}$ and all their factorisations.

By considering, for given m, k and for any N, colourings χ which map any factorisation $\overline{w}$ of $w \in A\binom{N}{k}$ into its type $\tau(\overline{w})$, we see that Theorem C is indeed the best possible. In other words, the type $\tau(\overline{w})$ of a factorisation is a discernable, and is, moreover, the only discernable under substitutions.

2. Proof of Theorem C

The proof of Theorem C will eventually follow from an iterated use of the main result in [1] (see 2.4). For this purpose, we first introduce a number of constructions and suitable bijections in Sections 2.1–2.3:

2.1. Let w be a k-parameter word over a finite alphabet A and $\pi(w) = \lambda_{i_1} \ldots \lambda_{i_l}$ the pattern associated with w. There is a factorisation $(v_1, \ldots, v_l)$ of w, unique up to the distribution of the alphabet letters, such that each v_j contains occurrences of exactly one parameter and that is λ_{i_j}. We write $\phi(w)$ for the word obtained from w by replacing the occurrences of λ_{i_j} in v_j by λ_j. Note that $\phi(w)$ is ascending and that w can be recovered from $\phi(w)$ if $\pi(w)$ is known. Also note that if $f \in A^{<}\binom{N}{m}$ and $w \in A\binom{m}{k}$ then $\phi(f \circ w) = f \circ (\phi(w))$.

For a factorisation $\overline{w} = (w_1, \ldots, w_s)$ of a parameter word w with associated pattern $\pi(w)$, we define $\phi(\overline{w})$ to be the factorisation

$$(\phi_{\pi(w)}(w_1), \ldots, \phi_{\pi(w)}(w_s))$$

of $\phi(w)$ by the requirement $|\phi_{\pi(w)}(w_i)| = |w_i|$. For example, if $w = 1\lambda_1\lambda_2\lambda_1\lambda_1\lambda_2\lambda_3\lambda_3\lambda_2\lambda_1$ and $\overline{w} = (1\lambda_1\lambda_2\lambda_1, \lambda_1\lambda_2\lambda_3\lambda_3\lambda_2\lambda_1)$ then $\pi(w) = \lambda_1\lambda_2\lambda_1\lambda_2\lambda_3\lambda_2\lambda_1$, $\phi(w) = 1\lambda_1\lambda_2\lambda_3\lambda_3\lambda_4\lambda_5\lambda_5\lambda_6\lambda_7$ and $\phi(\overline{w}) = (1\lambda_1\lambda_2\lambda_3, \lambda_3\lambda_4\lambda_5\lambda_5\lambda_6\lambda_7)$.

Note that if $f \in A^{<}\binom{N}{m}$ and $w \in A\binom{m}{k}$ and $\overline{w} = (w_1, \ldots, w_s)$ then

$$\phi(f \star \overline{w}) = f \star \phi(\overline{w}).$$

Indeed, $f\star\phi(\overline{w})=(f_1\circ\phi_{\pi(w)}(w_1),\dots,f_s\circ\phi_{\pi(w)}(w_s))$ where $(f_1,\dots,f_s)$ is a factorisation of f such that the first letter of f_i is $\lambda_{v(i)}$ where for $i\geqslant 2$,

$$\begin{aligned} v(i) &= |\phi_{\pi(w)}(w_1)| + \cdots + |\phi_{\pi(w)}(w_{i-1})| + 1 \\ &= |w_1| + \cdots + |w_{i-1}| + 1. \end{aligned}$$

This means that $f\star\overline{w}=(f_1\circ w_1,\dots,f_s\circ w_s)$ and since $f\star\overline{w}$ is a factorisation of $f\circ w$ and $\pi(f\circ w)=\pi(w)$, we have

$$\begin{aligned} \phi(f\star\overline{w}) &= (\phi_{\pi(w)}(f_1\circ w_1),\dots,\phi_{\pi(w)}(f_s\circ w_s)) \\ &= (f_1\circ\phi_{\pi(w)}(w_1),\dots,f_s\circ\phi_{\pi(w)}(w_s)) \\ &= f\star\phi(\overline{w}). \end{aligned}$$

2.2. A word π over the set of parameters $\{\lambda_1,\dots,\lambda_k\}$ is called a *pattern* if no two consecutive letters in π are the same.

Let A be a finite alphabet, $\{\lambda_1,\dots,\lambda_k\}$ a set of parameters and

$$\tau=(\pi_1,(x_2,\pi_2),\dots,(x_s,\pi_s))$$

an s-tuple with $\pi_1,\dots,\pi_s$ patterns over $\{\lambda_1,\dots,\lambda_k\}$ and x_i either in A or the first parameter in π_i. Then τ is called a *factorisation-type* (resp. *ascending factorisation-type*) over A and $\{\lambda_1,\dots,\lambda_k\}$, if the pattern π of the word $\pi_1\dots\pi_s$ is a parameter word (resp. ascending parameter word) over $\{\lambda_1,\dots,\lambda_k\}$. Let $\mathscr{F}_A\binom{n}{k}$ denote the set of all factorisations $\overline{w}$ of all words $w\in A\binom{n}{k}$ and $\mathscr{F}_A\binom{n}{k}_\tau$ all those of type τ. Note that if τ is an ascending factorisation-type then all the factorisations in $\mathscr{F}_A\binom{n}{k}_\tau$ are the factorisations of ascending parameter words.

For any factorisation-type τ over A and $\{\lambda_1,\dots,\lambda_k\}$ the s-tuple,

$$\tau_\phi := (\phi_\pi(\pi_1),(x_2',\phi_\pi(\pi_2)),\dots,(x_s',\phi_\pi(\pi_s)))$$

with

$$x_i' = \begin{cases} x_i & \text{if } x_i\in A, \\ \text{the first letter of } \phi_\pi(\pi_i) & \text{otherwise} \end{cases}$$

is an ascending factorisation-type over A and $\{\lambda_1,\dots,\lambda_l\}$ where l is the length of the pattern π of $\pi_1\pi_2\dots\pi_s$. It is clear that if $\overline{w}=(w_1,\dots,w_s)$ is a factorisation of a word $w\in A\binom{m}{k}$, then $\tau(\phi(\overline{w}))=\tau_\phi$ since

$$\pi(\phi_{\pi(w)}(w_n))=\phi_{\pi(w)}(\pi(w_n))$$

and the first letter of $\phi_{\pi(w)}(w_n)$ is the first letter of w_n if it is not a parameter.

Let $\tau=(\pi_1,(x_2,\pi_2),\dots,(x_s,\pi_s))$ be any factorisation-type over A and $\{\lambda_1,\dots,\lambda_k\}$ and let l be the length of the pattern of $\pi_1\pi_2\dots\pi_s$. Then for a given natural number n, the rule which associates with every $\overline{w}\in\mathscr{F}_A\binom{n}{k}_\tau$, the factorisation $\phi(\overline{w})\in\mathscr{F}_A\binom{n}{l}_{\tau_\phi}$ is a bijection

$$\phi:\mathscr{F}_A\binom{n}{k}_\tau\to\mathscr{F}_A\binom{n}{l}_{\tau_\phi}.$$

2.3. In the sequel, A is a fixed finite alphabet. For natural numbers k, m, n, r and a factorisation-type τ over A and $\{\lambda_1, \ldots, \lambda_k\}$ we write:

$$\begin{array}{c} m \cdots\cdots n \\ \bigtriangledown_{\tau}\, r \\ k \end{array} \tag{\#}$$

if for every r-colouring χ of $\mathscr{F}_A\binom{n}{k}_\tau$ there exists a monochromatic $f \in A^{<}\binom{n}{m}$, i.e. $\chi(f \star \overline{u}) = \chi(f \star \overline{v})$ for every $\overline{u}$, $\overline{v} \in \mathscr{F}_A\binom{m}{k}_\tau$.

Let $\tau = (\pi_1, (x_2, \pi_2), \ldots, (x_s, \pi_s))$ be any factorisation-type over A and $\{\lambda_1, \ldots, \lambda_k\}$ and π the pattern of $\pi_1 \pi_2 \ldots \pi_s$. We show that if $|\pi| = l$ and ϕ and τ_ϕ are as defined in Sections 2.1 and 2.2, then

$$\begin{array}{c} m \cdots\cdots n \\ \bigtriangledown_{\tau_\phi}\, r \\ l \end{array} \quad \text{implies} \quad \begin{array}{c} m \cdots\cdots n \\ \bigtriangledown_{\tau}\, r \\ k \end{array} \tag{$*$}$$

For a given r-colouring χ of $\mathscr{F}_A\binom{n}{k}_\tau$ define an r-colouring χ' of $\mathscr{F}_A\binom{n}{l}_{\tau_\phi}$ by

$$\chi'(\overline{u}) := \chi(\phi^{-1}(\overline{u})) \quad \text{for every } \overline{u} \in \mathscr{F}_A \binom{n}{l}_{\tau_\phi}.$$

By the hypothesis in $(*)$, there exists an $f \in A^{<}\binom{n}{m}$ which is monochromatic with respect to χ'. Consequently by (#), for every $\overline{w} \in \mathscr{F}_A\binom{m}{k}_\tau$,

$$\chi(f \star \overline{w}) = \chi'(\phi(f \star \overline{w})) = \chi'(f \star \phi(\overline{w})),$$

showing that f is also monochromatic with respect to χ, and that $(*)$ holds.

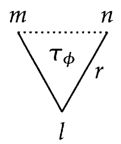
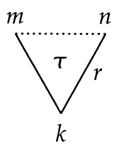
The following Lemma is based on Theorem B of [1].

Lemma. *For natural numbers k, m and r, any finite alphabet A and any factorisation-type τ over A and $\{\lambda_1, \ldots, \lambda_k\}$, there exists a natural number n that depends primitive recursively on k, m, r and $|A|$ such that*

$$\begin{array}{c} m \cdots\cdots n \\ \bigtriangledown_{\tau}\, r \\ k \end{array}$$

holds.

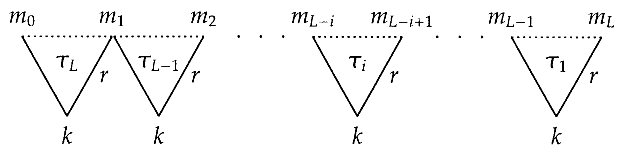
Proof. Let $\tau = (\pi_1, (x_2, \pi_2), \ldots, (x_s, \pi_s))$ be any factorisation-type over A and $\{\lambda_1, \ldots, \lambda_k\}$ and let l be the length of the pattern of $\pi_1 \pi_2 \ldots \pi_s$. Let ϕ and τ_ϕ be as defined in Sections 2.1 and 2.2. Then τ_ϕ is an ascending factorisation-type and therefore all factorisations of type τ_ϕ are factorisations of ascending parameter words. It now follows from Theorem B of [1] that there exists a primitive recursive function $n(k, m, r, |A|)$ such that for $n = n(k, m, r, |A|)$,


holds. By $(*)$,


now holds and we are done. □

2.4. We now conclude the proof of Theorem C by a suitable iteration of the lemma in Section 2.3.

Let $\tau_1, \ldots, \tau_L$ be all the different factorisation-types of the factorisations in $\mathscr{F}_A\binom{m}{k}$. Set $m_0 = m$. We define the natural numbers $m_1, \ldots, m_L$ iteratively such that



Set $N = m_L$ and let χ be any r-colouring of $\mathscr{F}_A\binom{N}{k}$.

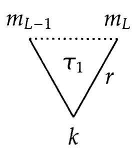
We prove that the following statement is true for every $i = 2, \ldots, L+1$.

$S(i-1)$: *There exists an ascending word* $f_{i-1} \in A^{<}\binom{m_L}{m_{L-i+1}}$ *such that for every*

$$\bar{u} \in \bigcup_{j=1}^{i-1} \mathscr{F}_A \binom{m_{L-i+1}}{k}_{\tau_j}$$

the colour $\chi(f_{i-1} \star \bar{u})$ *depends on the type of* $\bar{u}$ *only.*

To show that $S(1)$ holds, let χ_1 be the restriction of χ to the factorisations in $\mathscr{F}_A\binom{m_L}{k}$ of type τ_1, i.e. $\mathscr{F}_A\binom{m_L}{k}_{\tau_1}$. Then by

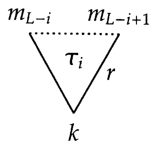


there exists a χ_1-monochromatic $f_1 = f_1' \in A^{<}\binom{m_L}{m_{L-1}}$, i.e. for every $\bar{u}_1, \bar{u}_2 \in \mathscr{F}_A\binom{m_{L-1}}{k}_{\tau_1}$, we have $f_1 \star \bar{u}_1$, $f_1 \star \bar{u}_2 \in \mathscr{F}_A\binom{m_L}{k}_{\tau_1}$ and $\chi(f_1 \star \bar{u}_1) = \chi(f_1 \star \bar{u}_2)$.

Now assume inductively that $S(i-1)$ holds. To show that $S(i)$ holds, let χ_i be the r-colouring defined by

$$\chi_i(\bar{u}) = \chi(f_{i-1} \star \bar{u})$$

for every $\overline{u} \in \mathscr{F}_A\binom{m_{L-i+1}}{k}_{\tau_i}$. Then by



there exists a χ_i-monochromatic parameter word $f_i' \in A^{<}\binom{m_{L-i+1}}{m_{L-i}}$. Furthermore, $f_i' \star \overline{u} \in \mathscr{F}_A\binom{m_{L-i+1}}{k}_{\tau_j}$ for every $\overline{u} \in \mathscr{F}_A\binom{m_{L-i}}{k}_{\tau_j}$ and $j=1,\ldots,i-1$. The statement $S(i)$ now holds for $f_i = f_{i-1} \circ f_i'$.

Since $S(L)$ holds, the truth of Theorem C for the given colouring χ is witnessed by $f = f_L$.

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